

PORTFOLIO RISK AND DEPENDENCE MODELING: APPLICATION OF FACTOR AND COPULA MODELS

Arsalan Azamighaimasi
Wuhan University of technology, china

Abstract

We consider portfolio credit risk modeling with a focus on two approaches, the factor model, and the copula model. While other models have received greater scrutiny, both factor and copula models have received little attention although these are appropriate for rating-based portfolio risk analysis. We review the two models with emphasis on the joint default probability. The copula function describes the dependence structure of a multivariate random variable. In this paper, it is used as a practical to simulation of generate portfolio with different copula, we only use Gaussian and t-copula case. And we generate portfolio default distributions and study the sensitivity of commonly used risk measures with respect to the approach in modeling the dependence structure of the portfolio.

Key Words: Gaussian copula, Factor model, Copula model

JEL Classification: C15, C38

1. Introduction

There is a need to understand components of portfolio risk and their interaction. The Basel Committee for Banking Supervision in its Basel proposed (BIS, 2001) to develop an appropriate framework for a global financial regulation system. Several portfolio credit risk models developed in the industry have been made public since then. Examples are: CreditMetrics (Gupton et al., 1997), CreditRisk+ (Credit Suisse Financial Products, 1997) and Credit Portfolio View (Wilson 1997a; 1997b). Others systems remain proprietary, such as KMV's Portfolio Manager (Kealhofer, 1996). Although the models appear quite different on the surface, recent

theoretical work has shown an underlying mathematical equivalence among them (Gordy, 2000; and Koyluoglu and Hickman, 1998).

The credit portfolio models to obtain portfolio loss distributions, which are statistical models, can be classified as based on credit rating systems; See Crouhy et al. (2001) for exact description and discussion of the various models. Frey and McNeil (2001) study the mathematical properties of the models and consider the modeling of dependent defaults in large credit portfolios using latent variable models and mixture models. Crouhy et al. (2000) compared and reviewed models on benchmark portfolio using credit migration approach, the structural approach, the actuarial approach, and McKinsey approach. However, few studies have attempted to investigate aspects of portfolio risk based on rating-based credit risk models. Gordy (2000) offered a comparative anatomy of two especially influential benchmarks for credit risk models, the Risk Metrics Group's Credit Metrics and Credit Suisse Financial Product's **CreditRisk⁺**. Kiesel et al. (1999) employ a mark-to-market model and stress the importance of stochastic changes in credit spreads associated with market values, an aspect also highlighted in Hirtle et al. (2001).

The aim of this paper is to contribute to the understanding of the performance of rating-based credit portfolio models, long ignored in the field. We apply a default-mode model to assess the effect of changing dependence structure within the portfolio. First, in the ensuing section, we discuss about the copula model as one of the dependency approaches within the portfolio. Second, we describe a factor model by focusing on the effects of default dependence model within the portfolio. Finally, in the penultimate section, we simulated types of copula model with different degree of freedom within the portfolio.

2. Copula Modelling

An overview of basic copula uses in structural systems and models is provided in this section. Copulas provide a natural way to study and measure dependence between random variables. Suppose we have specified a portfolio of n obligors, with default times $\tau_1, \tau_2, \dots, \tau_n$. The variable

of default of obligor $i, i = 1, 2, \dots, n$, at time t , is denoted as $Y_i(t) := \mathbf{1}_{\{\tau_i \leq t\}}$. The probability space is $(\Omega, \mathcal{H}, \mathbf{P})$. This space has filtration $\mathbb{G}_i := \{\mathcal{G}_{it}; t \geq 0\}$:

$$\mathcal{G}_{it} := \sigma(Y_i(u); 0 \leq u \leq t) \quad (1)$$

For the joint default probability at time t , evaluated at time 0, as $F(t)$

$$F(t) := P(Y_1(t) = 1, Y_2(t) = 1, \dots, Y_n(t) = 1 | \mathcal{G}_0) \quad (2)$$

The survival property as $s(t)$

$$s(t) := P(Y_1(t) = 0, Y_2(t) = 0, \dots, Y_n(t) = 0 | \mathcal{G}_0) \quad (3)$$

We take for granted the copula definition as a joint distribution function with uniform margins, which implies that \mathcal{C} and take for granted the fundamental Sklar's theorem, in terms of a copula \mathcal{C} and the marginal distribution functions $F_i(t), i = 1, 2, \dots, n$:

$$F(t) = \mathcal{C}(F_1(t), F_2(t), \dots, F_n(t)) \quad (4)$$

The joint survival probability $s(t)$ with survival copula, $\check{\mathcal{C}}$ and the marginal survival functions $S_i(t) := 1 - F_i(t)$:

$$s(t) = \check{\mathcal{C}}(S_1(t), S_2(t), \dots, S_n(t)) \quad (5)$$

Factor copula \mathcal{C}^\perp is,

$$\mathcal{C}^\perp(u_1, \dots, u_n) = u_1 \times u_2 \times \dots \times u_n. \quad (6)$$

In the credit risk case, since the variables τ_i are default time, the copula represents default dependence. It is denoted as \mathcal{C}^τ ,

$$F(t) = \mathcal{C}^\tau(F_1(t), F_2(t), \dots, F_n(t)) \quad (7)$$

$$s(t) = \check{\mathcal{C}}^\tau(S_1(t), S_2(t), \dots, S_n(t)) \quad (8)$$

According to Merton model (1974) if default of firm (i) occurs, the values of asset or values of shares cross from barrier line of outstanding debt at debt maturity. Default is occurred when the firm's asset value $V_i(t)$ falls to the liability one, $K_i(t)$, the time of default is:

$$\tau_i = \begin{cases} t & P(V_i(t) \leq K_i(t)) \\ +\infty & P(V_i(t) > K_i(t)) \end{cases} \quad (9)$$

The default probability at time t is,

$$F_i(t) = P(V_i(t) \leq K_i(t)) \quad (10)$$

The marginal default probability can be easily computed to be

$$F_i(t) = \Phi(d_{2i}(t)) \quad (11)$$

Then,

$$d_{2i}(t) = \frac{\ln\left(\frac{V_i(t)}{K_i(t)}\right) + \left(m - \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}} \quad (12)$$

And m is the instantaneous return on assets, which equates the riskless rate r under the risk neutral measure. The joint default probability of n assets is

$$F_1(t) = P(V_1(t) \leq K_1(t), \dots, V_n(t) \leq K_n(t); \mathcal{G}_0) = \Phi_R(-d_{21}(t), \dots, -d_{2n}(t)) \quad (13)$$

Where, Φ_R is the distribution function of a standard normal vector with correlation matrix R . the marginal default probabilities is follows

$$F(t) = \Phi_R(\Phi^{-1}(F_1(t)), \dots, \Phi^{-1}(F_n(t))) \quad (14)$$

To study the effect of different copula on default correlation, we use the following examples of copula (further details on these copula can be found in Embrechts et al., 2001).

(i) Gaussian copula:

$$C_R(u_1, \dots, u_n) = \Phi_R^n(\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_n)) \quad (15)$$

Where, Φ_R^n denotes the joint distribution function of the n - variety normal with linear correlation matrix R , and Φ^{-1} the inverse of the distribution function of the univariate standard normal.

(ii) A student t copula:

$$c_{v,R}(u_1, \dots, u_n) = t_{v,R}^n(t_v^{-1}(u_1), \dots, t_v^{-1}(u_n)) \quad (16)$$

Where $t_{v,R}$ is the standardized multivariate Student's t distribution, with correlation matrix R and v degrees of freedom, While t_v^{-1} is the inverse of the corresponding margin.

Gumbel copula:

$$C_\theta(u_1, \dots, u_n) = \exp\{-[(-\log u_1)^\theta + \dots + (-\log u_n)^\theta]^{1/\theta}\} \quad (17)$$

Where, $\theta \in [1, \infty)$. This class of copula is a sub-class of the class of Archimedean copula.

According to the table[1], joint default probabilities of two obligors are represented through three types of obligors with individual default probabilities corresponding to rating classes.as you will see that t and Gumbel copula have higher joint default probabilities than the Gaussian copula.

The joint default probabilities of two Obligors are represented through three types of obligors with individual default probabilities corresponding to rating classes.

Table 1: Copula and default probability

copula	Default probability		
	Class A($\times 10^{-6}$)	Class B($\times 10^{-4}$)	Class C($\times 10^{-4}$)
* [-2pt]Gaussian	6.89	3.38	52.45
C_{10}^t	46.55	7.88	71.03
C_4^t	134.80	15.35	97.96
Gumbel, C_2	57.20	14.84	144.56
Gumbel, C_4	270.60	41.84	283.67

3. Factor Modelling

Another popular approach to default modeling allows us to switch to the so called product copula. The reduction technique, which is widely adopted for the evaluation of losses in high-dimensional portfolios, with hundreds of obligors (see for instance Laurent and Gregory (2003)), is the standard approach of (linear) factorization, or transformation into a Bernoulli factor model.

In the typical portfolio analysis the vector \mathbf{V} is embedded in a factor model, which allows for easy analysis of correlation, the typical measure of dependence. We assume that the underlying variables V_j are driven by a vector of common factors.

$$V_j = \sum_{i=1}^p a_{ij} Z_i + \sigma_j \epsilon_j, j = 1, \dots, n \quad Z \sim N(0, \Sigma) \quad (18)$$

Where \mathbf{p} is dimensional normal vector, and ϵ is independent normally distributed random variables. Here a_{ij} is obligor j to factor i , i.e. the so-called factor loading and σ_j is volatility of the risk contribution. The default indicators $Y_j(t)$ of the n obligor are independent Bernoulli variables, with probability:

$$Y_j(t) = \begin{cases} 1 & V_j \leq K_j \\ 0 & \text{otherwise} \end{cases} \quad (19)$$

Where K_j is cut-off point for default obligor j . The individual default probabilities are,

$$F_j = P(Y_j = 1) = P(V_j \leq K_j), \quad (20)$$

And the joint default probability is,

$$F_{ij} = P(Y_i = 1, Y_j = 1) = P(V_i \leq K_i, V_j \leq K_j). \quad (21)$$

If we denote by $\rho_{ij} = \text{Corr}(V_i, V_j)$ the correlation of the underlying latent variables and by $\rho_{ij}^K = \text{Corr}(Y_i, Y_j)$ the default correlation of obligors i and j , then we obtain the default correlation formula

$$\rho_{ij}^K = \frac{F_{ij} - F_i F_j}{\sqrt{F_i F_j (1 - F_i)(1 - F_j)}} \quad (22)$$

Under assumption above, we obtain the joint default probability,

$$F_{ij} = \int_{-\infty}^{K_i} \int_{-\infty}^{K_j} \varphi(u, v; \rho_{ij}) du dv, \quad (23)$$

Where $\varphi(u, v; \rho)$ is bivariate normal density with correlation coefficient ρ .

4. Simulation Results of Copula Model

Here, we want to generate portfolios with given marginal and the above copula. we only use Gauss and t – copula case . We looking for random sample generation for this mean we obtain the generation of an n -variety normal with liner correlation matrix $R, (x_1, \dots, x_n)^T \sim N(0, R)$, to take realizations from a Gaussian copula we simply have to transform the marginal:

- Set $u_i = \Phi(x_i), \quad i = 1, \dots, n$
- $(u_1, \dots, u_n)^T \sim C_R^{GAUSS}$

To generate random varieties from the t –copula $C_{v,R}^t$ we assume the random vector X act the stochastic process

$$X = \mu + \sqrt{\frac{v}{Z}} Y \quad (\text{in distribution}), \quad (24)$$

With

$$\mu \in \mathbb{R}^n, \quad Z \sim \chi_v^2 \quad \text{and} \quad Y \sim N(0, \Sigma)$$

Where Z and Y are independent, and then X is t_v distributed with mean μ and covariance matrix $(\frac{v}{v-2}\Sigma)$. we assume $v > 2$, while the stochastic process is still valid the parameters has to change for $v \leq 2$. We will have algorithm (this is algorithm in Embrechts et al. (2001)):

- Set $x = \sqrt{\frac{v}{z}}$
- Set $u_i = t_v(x_i), \quad i = 1, \dots, n$
- $(u_1, \dots, u_n)^T \sim C_{v,R}^t$.

We can replace the u_i with $\Phi^{-1}(u_i)$ in order to have multivariate distribution with t -copula and normal marginal, to obtain the t -copula $C_{v,R}^t$.

Figure 1 shows three simulation results with 1000, 500, and 50 observations from a multivariate normal distribution. As you see the represents tree types of observations from a multivariate normal distribution with mean vector μ and covariance matrix. The figure 2 shows to computes a scatterplot of a normal sample and in a second plot the contour ellipses for $\mu = \#$ (3, 2) and $\sigma = \#$ (1,-1.5) \sim $\#$ (-1.5, 4) with different observations.

Figure1: Simulation results from samples of 1,000, 500 and 50 observations

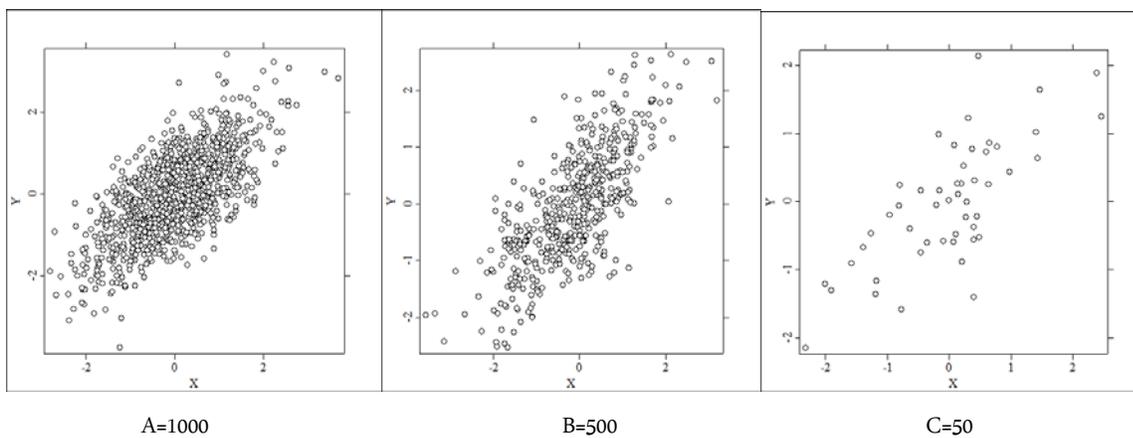
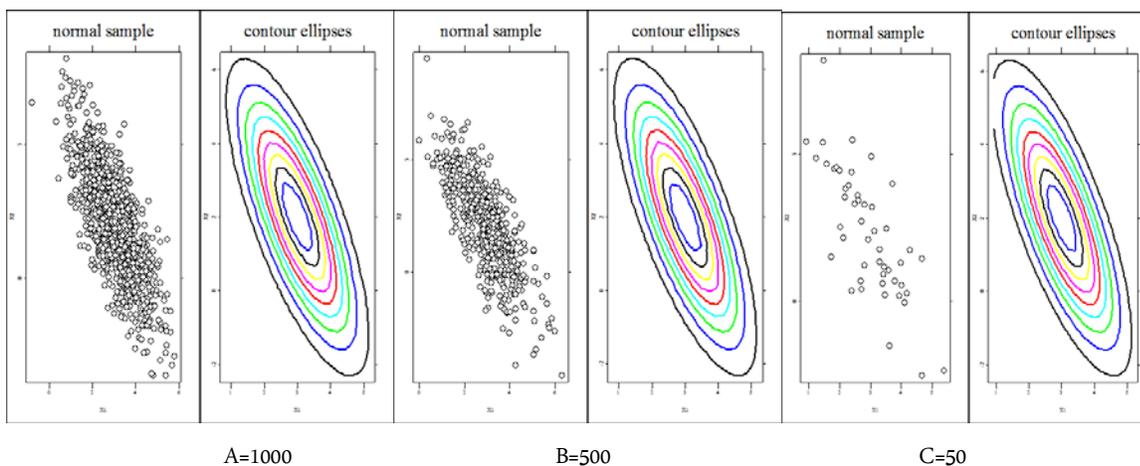


Figure 2: Scatter plots of normal sample and second plot of the contour ellipses



Further analyses of the same data are plotted in Figure 2. These are scatterplots of a normal sample. In an adjacent plot next to each sample, we present a second plot as the contour ellipses for $\mu = \#(3, 2)$ and $\sigma = \#(1, -1.5) \sim \#(-1.5, 4)$ with different size.

4.1 Portfolio

For our first simulation exercise, we assume that the underlying variables V_j are normally distributed within a single factor framework, i.e. $p = 1$ and α_{j1} in formula as follow:

$$V_j = \sum_{i=1}^p \alpha_{ij} Z_i + \sigma_j \epsilon_j, j = 1, \dots, n \quad (25)$$

They are constant and are chosen so that the correlation for the underlying latent variables V_j is $\rho = 0.2$ (Kiesel et al., 1999). Note that we use three rating classes, named A, B, and C with default probabilities 0.005, 0.05, and 0.15 roughly corresponding to default probabilities from standard rating classes (Ong, 1999). To generate different degrees of tail correlation, we link the individual assets together using a Gaussian, a t_{10} and a t_4 -copula.

The information in table 2, 3 and 4 represent the effect tail-dependence has on the high quintiles of highly-rated portfolios at different quintiles: Table 2 is for 99 percentile.

Table2: Effect of normal copula with default probability set at 0.005

Portfolio	Copula	Mean	variance	$\alpha : 0.995$	$\alpha : 0.999$
A=1000	normal	0.115	0.13391	1	2
A=500	normal	0.106	0.119	1	1
A=50	normal	0.18	0.19143	1	2
B=1000	normal	0.99	1.8277	4	6
B=500	normal	1.038	1.8442	4	6
B=50	normal	1.18	2.3955	4	6
C=1000	normal	3.029	7.0953	8	11
C=500	normal	2.998	6.9078	8	11
C=	normal	3.1	7.3163	9	10
50					

The t_4 copula is more than three-times larger than the corresponding quintile for the Gaussian copula. The same effect can be observed for lower rated portfolios although not quite with a similar magnitude.

Table3: effect of t_{10} – copula with default probability 0.05

Portfolio	Copula	Mean	variance	$\alpha = 0.995$	$\alpha = 0.999$
A=1000	t_{10}	0.101	0.26907	1	2
A=500	t_{10}	0.098	0.15671	1	2
A=50	t_{10}	0.14	0.36776	1	4
B=1000	t_{10}	0.963	2.38	4	6
B=500	t_{10}	0.994	2.1984	4	6
B=50	t_{10}	1.06	2.9147	4	9
C=1000	t_{10}	3.008	7.9799	9	11
C=500	t_{10}	3.05	7.9474	9	12
C=50	t_{10}	3.42	8.9016	9	11

We assume the second factor, i.e. $p = 2$ in (4), for a sub-portfolio of 100 obligors increasing the correlation of the latent variables V_j within the sub-portfolio to 0.5

Table4: effect of t_4 – copula with default probability 0.15

Portfolio	Copula	Mean	variance	$\alpha = 0.995$	$\alpha = 0.999$
A=1000	t_4	0.088	0.39665	0	2
A=500	t_4	0.084	0.24543	0	2
A=50	t_4	0.22	2.42	0	11
B=1000	t_4	0.924	3.1454	5	9
B=500	t_4	1	3.0261	7	5
B=50	t_4	1.02	3.5302	4	11
C=1000	t_4	2.997	9.5860	10	12
C=500	t_4	3.028	9.0213	9	13
C=50	t_4	3.34	9.2086	9	12

Table 5: the effect of correlation cluster with default probability 0.005

portfolio	copula	First subportfolio	Second subportfolio	mean	variance	0.995	0.999
A=1000	normal	100	150	1.237	6.8447	5	13
A=500	normal	50	75	0.6	1.6433	2	7
A=50	normal	20	30	0.24	0.47184	1	4
B=1000	normal	100	150	12.723	204.41	41	71
B=500	normal	50	75	6.198	47.951	20	33
B=50	normal	20	30	2.58	7.3506	10	11
C=1000	normal	100	150	37.972	871.43	96	132
C=500	normal	50	75	18.832	200.1	49	63
C=50	normal	20	30	7.74	30.36	20	23

Table 6: the effect of correlation cluster with default probability 0.05

portfolio	copula	First subportfolio	Second subportfolio	mean	variance	0.995	0.999
A=1000	t_{10}	100	150	1.451	27.335	7	28
A=500	t_{10}	50	75	0.644	6.7668	3	11
A=50	t_{10}	20	30	0.2	0.32653	1	3
B=1000	t_{10}	100	150	11.76	299.29	52	83
B=500	t_{10}	50	75	6.28	85.605	24	44
B=50	t_{10}	20	30	2.32	11.365	10	17
C=1000	t_{10}	100	150	38.24	1104.7	105	148
C=500	t_{10}	50	75	18.638	263.7	52	75
C=50	t_{10}	20	30	7.5	31.235	17	24

Table 7: the effect of correlation cluster with default probability 0.15

portfolio	copula	First subportfolio	Second subportfolio	mean	variance	0.995	0.999
A=1000	$t_{\frac{1}{2}}$	100	150	1.635	70.278	7	42
A=500	$t_{\frac{1}{2}}$	50	75	0.682	14.554	3	21
A=50	$t_{\frac{1}{2}}$	20	30	0.36	2.1943	1	10
B=1000	$t_{\frac{1}{2}}$	100	150	13.385	592.25	65	128
B=500	$t_{\frac{1}{2}}$	50	75	6.266	132.82	28	61
B=50	$t_{\frac{1}{2}}$	20	30	2.26	16.074	13	18
C=1000	$t_{\frac{1}{2}}$	100	150	38.465	1395	117	157
C=500	$t_{\frac{1}{2}}$	50	75	18.676	331.96	56	80
C=50	$t_{\frac{1}{2}}$	20	30	7.56	41.109	23	27

for this reasoning we want to show the effects of increased correlation within parts of the portfolio; we change the factor loading within parts of our portfolio. These results are shown in tables 7, 9 and 10.

As expected, the results in Tables 5, 6, 7 show increase in the quantiles due to the increased correlation within the portfolio. However, comparing the three tables we will see that the sensitivity of the portfolio loss quantiles is higher with regard to the underlying copula than to the correlation within the portfolio.

5. Conclusions

To investigate the riskiness of credit-risky portfolios is one of the big challenges in financial mathematics. An important thing for a model of credit-risky portfolios is the dependence structure of the underlying obligors. We studied two approaches, a factor structure, and the direct specification of a copula. We generated portfolio default distributions and studied the sensitivity

of commonly used risk measures with respect to the approaches in modeling the dependence structure of the portfolio using as a rating-based approach using copula mathematics.

The simulation results indicate that the degree of tail dependence of the underlying copula plays a major role. That is identified as a credit risk. The copula modeling links the underlying variables together, which is of crucial importance especially for portfolios of highly-rated obligors.

Author Information: Arsalan Azamighaimasi is a faculty member in the Department of Management, Wuhan University of Technology, China. He may be contacted at E-mail: Arsalan.azami2011@gmail.com.

References

BIS (2001), "Overview of the new Basel capital accord," Basel Committee on Banking Supervision.

Caouette, J., Altman, E. and Narayanan, P., (1998). "Managing Credit Risk, The Next Great Financial Challenge," Wiley Frontiers in Finance, Vol. Wiley Frontiers in Finance, Wiley & Sons, Inc, New York.

M. Carey, (2000). "Dimensions of credit risk and their relationship to economic capital requirements.," Federal Reserve Board.

M. Gordy, (1997). "A comparative anatomy of credit risk models," *Journal of Banking and Finance*, vol. 24, pp. 119-149, 2000J. Morgan, "Creditmetrics," JP Morgan, New York.

Crouhy, M., Galai, D., and Mark, R., (2001). Risk management, McGraw Hill.

Crouhy, M., Galai, D. and Mark, R, (2000). "A comparative analysis of current credit risk models," *ournal of Banking and Finance*, Vols. 24(1-2), pp. 59-117.

Embrechts, P., Lindskog, F. and McNeil, A, (2001). "Modelling dependence with copulas and applications to risk management," ETH Zürich.

Frey, R. and McNeil, A, (2001). "Modelling dependent defaults," ETH Zürich.

P. Jorion, (2000). Value at Risk, New York. : McGraw-Hill.

Kiesel, R., Perraudin, W. and Taylor, A, (1999). "The structure of credit risk," Birkbeck College.

Koyluoglu, H., and Hickmann, A., (1998). "A generalized framework for credit portfolio models," Wyman & Company,

Luciano, E., (2008). "Copula-based default dependence modelling: where do we stand?," in Nickell, P., Perraudin, W. and Varotto, S, "Ratings-versus equity-based credit risk models: An empirical investigation," unpublished Bank of England mimeo.

M. Ong, (1999). *Internal Credit Risk Models. Capital Allocation and Performance Measurement*, London: Risk Books.