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COMPENSATING BALANCE: A COMMENT

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Abstract

Compensating balances are deposits the borrowing firm keeps with the lending bank in non-interest bearing accounts on loans. It is well known that, when there is no compensating balance imposed, the effective cost of debt remains the same in the two different payment methods, full amortization method and bullet loan (bond) method. It has been experimentally shown that when a compensating balance is imposed, however, the respective effective costs of debt under the two payment methods become different and that the true cost of a fully-amortized loan is always greater than that of a bullet loan. This paper provides a mathematical proof that this is always true. It concludes that, whenever a bank imposes a compensating balance, the borrowing firm should prefer a bullet loan to a fully-amortized one if it wishes to avoid an ambush posed by such a compensating balance.

Keywords: Compensating balance, effective cost of debt, net present value, amortization, bullet payment method

1. Introduction

Compensating balances are deposits the borrower (firm) keeps with the lender (bank) in non-interest bearing accounts on the term loan. See Ross, Westerfield and Jaffe ([12]), Downes and Goodman ([4]), Davidson III ([3]). Historically, compensating balances have been frequently used as covenants in bank lending arrangements, as in Hagaman ([7]), Gallager ([6]) and Hingston ([8]). Under these arrangements, the borrowing firm is required to maintain a cash balance at the lending bank equal to a certain percentage of the loan balance during the life of the loan. See Booth and Chua ([1]). By leaving these funds with the lending bank without receiving interest, the borrowing firm increases the effective (true) interest the firm pays to the bank on a loan. See Kolodny, Seeley and Polakoff ([10]), Sealey and Heinkel ([13]), Lam and Boudreaux ([11]) and Burr ([2]).

It is well known that, when there is no compensating balance, the effective cost of debt remains the same in the two different, frequently-used payment methods, i.e., full amortization method and bullet loan (bond) method. When a compensating balance is imposed, however, the effective costs of debt will surely increase for both types of loans. An important question arises: will the effective costs for both types of a loan still be the same at a higher rate? If not, which one will involve a higher (or lower) effective borrowing cost? Answers to these questions are not obvious (at least to the authors). Moreover, we have not found a rigorous analysis of this issue in either any textbook or academic journal.

After setting the stage for discussion with a simple numerical example (Section II), we conduct a mathematical analysis of the patterns of the respective effective costs of debt when a compensating balance is imposed (Section III). Concluding comments are provided in Section IV. Appendix contains detailed proofs which supplement our results and an open question about the relation between compensating balance and loan period.

2. Full amortization vs. bullet loan

Consider a simple numerical example of borrowing $10,000 for three years at an annual quoted interest rate of 10% per year under two alternative ways, A1 and B1.

Under A1, the loan is fully amortized and the annual payment at the end of each year (ordinary annuity) is $4,021.15. The pattern of annual cash flows (for the borrowing firm) is +10,000, -4,021.15, -4,021.15, -4,021.15. Both the quoted interest rate and effective interest rate (internal rate of return) are 10% per year. Alternative B1 is a bullet (interest only) loan. The pattern of annual cash flows is +10,000, -1,000, -1,000, -11,000. Both the quoted interest rate and effective interest rate (internal rate of return or IRR) are also 10% per year. Now the lending bank imposes a compensating balance of, say, $2,000 on both A1 and B1. The borrowing firm deposits $2,000 at Year 0 (the beginning of the loan period) and gets it back from the bank at Year 3 (the end of the loan period). A1's pattern of cash flows has now become: +8,000, -4,021.15, -4,021.15, -2,021.15, and the effective interest rate increases to 13.88%. Let us call this new alternative A2. B1's pattern of cash flows has now become +8,000, -1,000, -1,000, -9,000 with the newly installed compensating balance, and the effective interest rate increases to 12.50%. Let us call this new alternative B2. Note that Under B2, the borrower is effectively borrowing $8,000 for three years at a quoted annual interest rate of 12.50%.

Why does the effective interest rate for A2 (13.88%) become greater than that for B2 (12.50%), even though an identical compensation balance is imposed on both alternatives? Is it because under A2 the borrower pays back periodically part of the loan principal whereas under B2 the borrower keeps the entire principal for three years? The answer to this question is no, because, for both baseline alternatives, A1 and B1, the effective interest rate is 10\% per year. Under B2, the firm effectively borrows $8,000 at an annual interest rate of 12.50% for three years. Under A2, the firm borrows the same effective amount of $8,000 for three years, but the amortization schedule is constructed for the loan principal of $10,000 (instead of $8,000) at an annual interest rate of 10% (instead of 12.50%). Why does the additional $2,000 used for the loan amortization more than compensate an interest rate reduction of 2.50% per year? The purpose of this paper is to provide a mathematical analysis that compares the effective costs of debt between the two alternatives for *all* possible combinations of the compensating balance and the loan period available in practice.

3. Effective cost of debt and compensating balance

Consider borrowing a principal P at a nominal interest rate of R and pay it back in n years at the end of each year. Depending on the payment method, we have the following four cases for cash flow chart and effective cost of debt.

A. *Full Amortization Method*

When an equal payment of A_n is made at the end of each time interval, we have the following cash flow chart.

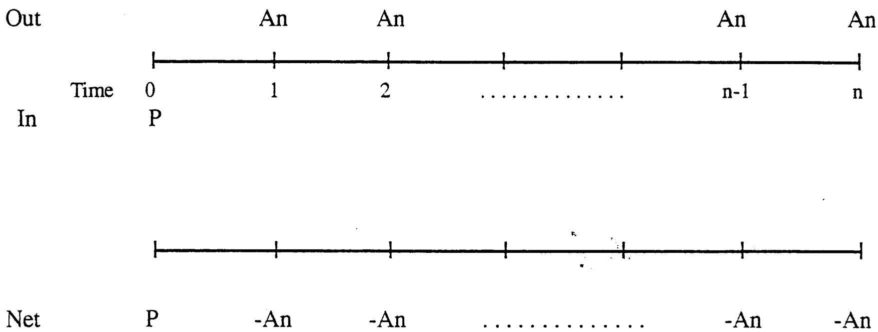

Fig. 1: Amortization without compensating balance

The net present value of this loan for the borrower is 0, hence $0 = P - A_n\left\{\frac{1}{1+R} + \frac{1}{(1+R)^2} + \dots \frac{1}{(1+R)^n}\right\}$ and this again implies

$$P = A_n\left\{\frac{1}{1+R} + \frac{1}{(1+R)^2} + \dots \frac{1}{(1+R)^n}\right\} = A_n \frac{1}{R}\left\{1 - \frac{1}{(1+R)^n}\right\} \qquad (1)$$

This amount $\frac{1}{R}\left\{1 - \frac{1}{(1+R)^n}\right\}$, called *the present value interest factor for an ordinary annuity* plays an important role in this paper, so we give it a formal definition of $f_n(R)$. Using this terminology, the end of year payment $A_n = P/f_n(R)$. The effective cost of debt r_A is the interest rate which makes the present value of cash-inflow and cash-outflow equal, and $r_A = R$ by our choice of A_n.

B. *Bullet Loan Method*

This is a typical payment schedule of bonds. An equal payment of PR (coupon) is made at the end of each time interval, and at the end of the loan period, the borrower pays back the principal P. The cash flow chart is the following.

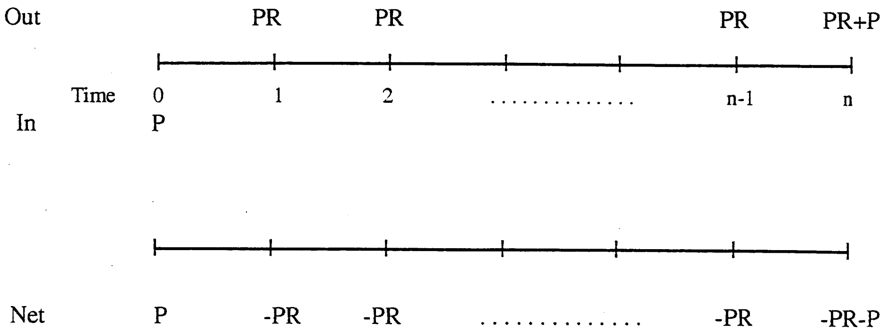

Fig.2: Bullet method without compensting balance

The geometric series makes the net present value zero, hence

$$0 = P - PR\left\{\frac{1}{1+R} + \frac{1}{(1+R)^2} + \dots + \frac{1}{(1+R)^n}\right\} - \frac{P}{(1+R)^n} \tag{2}$$

Again, by the construction of this payment method, the effective cost of debt r_B is the same as the nominal rate R.

Now, assume the lender (bank) requires a compensating balance X which will be returned to the borrower at the end of the loan period. No interest will be compounded and the borrower will receive only the original amount X. Theoretically, the compensating balance can be any number between 0 and the principal P, but in practice it is usually less than 30% of the principal. Throughout this paper, we assume the compensating balance is less than half of the principal, $X < P/2$. This choice of P/2 guarantees the uniqueness of the effective cost of debt (Appendix B). Our goal is to prove that there is only one effective cost of debt, r_A and r_B, respectively, for each payment method considered, and compare their sizes. Despite our assumption throughout this paper that all the payments are made at the end of each year, in practice, such payments are often made monthly. This means some payment methods can have positive net cash flows at the end of the loan period (recall that the cash flow is calculated in the borrower's position), causing more than one sign changes in the net cash flows. According to Descartes' Rule of Signs, this can cause multiple effective costs of debt, and our restriction $0 < X < P/2$ assures that this will not happen.

C. Full Amortization Method with Compensating Balance

Assuming a compensating balance X is imposed in case (a).

Out	X	An	An			An	An
Time	0	1	2	....		n-1	n
In	P						X
Net	P-X	-An	-An	....		-An	X-An

Fig. 3: Amortization with compensating balance

The end of year payment A_n remains the same and r_A is decided by

$$0 = (P - X) - A_n\left\{\frac{1}{1+r_A} + \frac{1}{(1+r_A)^2} + \ldots + \frac{1}{(1+r_A)^n}\right\} + \frac{X}{(1+r_A)^n}$$

or equivalently

$$(P - X) + \frac{X}{(1+r_A)^n} = A_n f_n(r_A) \text{ where } A_n = \frac{P}{f_n(R)} \qquad (3)$$

D. Bullet Loan Method with Compensating Balance

Finally, we have a bullet loan method with a compensating balance X imposed.

Out	X	PR	PR			PR	PR+P
Time	0	1	2	....		n-1	n
In	P						X
Net	P-X	PR	PR	....		-PR	PR-P+X

Fig. 4: Bullet method without compensating balance

r_B is decided by zero net present value of the cash flow,

$$0 = (P - X) - PR\left\{\frac{1}{1 + r_B} + \frac{1}{(1+r_B)^2} + \ldots + \frac{1}{(1+r_B)^n}\right\} - \frac{P}{(1+r_B)^n} + \frac{X}{(1+r_B)^n}$$

or equivalently

$$(P - X) - \frac{P}{(1+r_B)^n} + \frac{X}{(1+r_B)^n} = Pf_n(r_B) \tag{4}$$

The function representing the present value interest factor for an ordinary annuity $f_n = (x) = \frac{1}{x}\left\{1 - \frac{1}{(1+x)^n}\right\}$ is a decreasing function of x (Appendix A) and plays an important role throughout the paper.

Now we prove that the effective cost of debt r_A under full amortization method is always greater than the nominal interest rate R and r_B. We also show that under the bullet payment method, the effective cost of debt r_B depends only on the compensating balance X and is independent of the loan period n.

Theorem 1. $r_A > R$ and $r_B = \frac{PR}{P - R)}$

Proof. Rewriting equation (3) using $A_n = \frac{P}{f_n(R)}$,

$$(P - X)\left\{1 - \frac{1}{(1+r_A)^n}\right\} = \frac{P}{f_n(R)} f_n r_A \tag{5}$$

$\frac{1}{(1 + r_A)^n}$ lies between 0 and 1 for any positive interest rate , so the left hand side of (5) is less than P, hence $f_n(r_A) < f_n(R)$.

Because $f_n(x)$ is a decreasing function, r_A is greater than equal to R. The equality holds when X=0.

The left hand side of equation (4) is

$$(P - X) - (P - X)\frac{1}{(1+r_B)^n} = (P - X)\left\{1- \frac{1}{(1+r_B)^n}\right\},$$

so equation (2) becomes

$$(P - X)\left\{1 - \frac{1}{(1+r_B)^n}\right\} = PR\frac{1}{r_B}\left\{1 - \frac{1}{(1+r_B)^n}\right\} \tag{6}$$

therefore $P - X = \dfrac{PR}{r_B}$ implies $r_B = \dfrac{PR}{P - X}$.

Note that this theorem proves the uniqueness of r_B but not that of r_A. It only proves that r_A is greater than the original interest rate R. Next theorem is the main result of this paper that the effective cost of debt under full amortization method is also unique and always higher than that under bullet payment method.

To prove this theorem, we introduce a function which is derived from the net present value of an investment project. Consider a full amortization payment method set throughout the paper, then the net present value of the borrower is a function of the loan period n, the compensating balance X and the discount rate r. Define $NPV(n,X,r) = (P-X) - A_n f_n(r) + \dfrac{X}{(1 + r_B)^n}$ where $A_n = \dfrac{P}{f_n(R)}$.

When we fix n and X, NPV(n,X,r) becomes a function of r which we denote as $NPV_{n,X}(r)$. Likewise, when we fix X and r, NPV(n,X,r) becomes a function of n, and we denote this function as $NPV_{r,X}(n)$. In terms of NPV(n,X,r), r_A is a zero of the polynomial equation $NPV(n,X,r) = (P-X) - A_n f_n(r) + \dfrac{X}{(1 + r_B)^n} = 0$. As mentioned earlier in Section II, we assume that the compensating balance is less than half of the principal, $0 < X < \dfrac{P}{2}$ and this guarantees that $NPV_{n,X}(r)$ is increasing in r (Appendix B), hence the effective cost of debt r_A is unique.

Theorem 2. $r_A \leq r_B$. The equality holds when $r_A = r_B = R$, that is, when the compensating balance X=0.

Proof. Consider $NPV(n,X,r) = (P-X) - A_n f_n(r) + \dfrac{X}{(1 + r_B)^n}$. This is an increasing function of r which meets the vertical $NPV_{n,X}$-axis at

$$NPV_{n,X}(0) = (P-X) - A_n\left\{\frac{1}{1+0} + \frac{1}{(1+0)^n} + \ldots + \frac{1}{(1+0)^n}\right\} - \frac{X}{(1+0)^n} \tag{7}$$

$$= p - X - A_n n + X = P - nA_n$$

and the horizontal r-axis at r_A (by the definition of r_A).

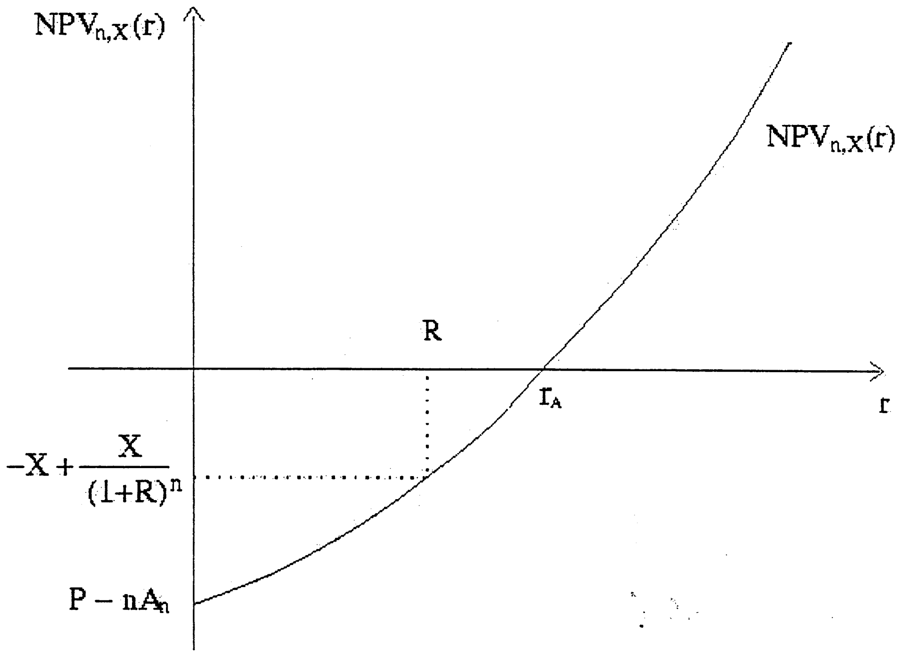

Fig. 5: Net present value as a function of r

$NPV_{n,X}(r)$ is increasing in r implies $r_A > r_B$, if and only if $NPV_{n,X}(r_B) < 0$. By the definition of r_B,

$$(P\text{-}X) - \frac{P-X}{(1+r_B)^n} - PRf_n(r_B) = 0 \qquad (8)$$

or equivalently,

$$(P\text{-}X) + \frac{X}{(1+r_B)^n} - PRf_n(r_B) + \frac{P}{(1+r_B)^n} \qquad (9)$$

Then

$$NPV_{n,X}(r_B) = (P\text{-}X) - A_n f_n(r_B) + \frac{X}{(1+r_B)_n} = PRf_n(r_B) + \frac{P}{(1+r_B)_n} - A_n f_n(r_B) \qquad (10)$$

$$= (PR-A_n)f_n(r_B) + \frac{P}{(1+r_B)^n} = \left\{ PR - \frac{PR(1+R)^n}{(1+R)^n-1} \right\} f_n(r_B) + \frac{P}{(1+r_B)^n}$$

$$= P\ 1 \left\{ \frac{-R}{(1+R)^n-1}\ \frac{1}{r_B}\ \frac{(1+r_B)^n-1}{(1+r_B)^n} + \frac{1}{(1+r_B)^n} \right\} = + \frac{P}{1+r_B)^n} + \left\{ \frac{-R\ (1+R_B)^n-1}{r_B\ (1+R_B)^n-1} \right\} + 1$$

$$= PR\left\{1 - \frac{(1+R)^n}{(1+R)^n - 1}\right\} f_n(r_B) + \frac{P}{(1+r_B)^n} = PR \frac{(1+R)^n - 1 - 1\ (1+R)^n}{(1+R)^n - 1} f_n(r_B) + \frac{P}{(1+r_B)^n}$$

Hence $NPV_{n,X}(r_B) < 0$ if it can be shown that

$$1 < \frac{R\ (1+r_B)^n - 1}{r_B\ (1+R)^n - 1} \quad \text{or equivalently} \quad \frac{r_B}{R} < \frac{R\ (1+r_B)^n - 1}{r_B\ (1+R)^n - 1} \tag{11}$$

Replace r_B by $\frac{PR}{P-X}$, then equation (11) becomes

$$\frac{\frac{PR}{P-X}}{R} < \frac{(1 + \frac{PR}{P-X})^n - 1}{(1+R)^n - 1}, \tag{12}$$

which is equivalent to

$$(P-X)(1+ \frac{PR}{P-X})^n - P\ (P-X) > P(1+R)^n - P. \tag{13}$$

Let $G(R) = (P-X)(1+ \frac{PR}{P-X})^n - (P-X) - P(1+R)^n + P$. Then

$$\begin{aligned} G'(R) &= (P-X)\ n\ (1+ \frac{PR}{P-X})^{n-1} - \frac{P}{P-X} - Pn(1+R)^{n-1} \\ &= Pn\{(1+ \frac{PR}{P-X})^{n-1} - (1+R)^{n-1}\} > 0 \quad \text{because} \\ &= 1+ \frac{PR}{P-X} = 1+ \frac{P}{P-X} R > +R. \end{aligned} \tag{14}$$

Moreover,

$$\begin{aligned} G(0) &= (P-X)n(1+0)^n - (P-X) - P(1+0)^n + P \\ &= (P-X) - (P-X) - P + P = 0 \end{aligned} \tag{15}$$

$G'(R) > 0$ for all $R > 0$ and $G(0) > 0$ implies $G(R) > 0$ for all $R > 0$, hence equation (13) has been proved. Because $F(R) > 0$ for all $R > 0$, hence $NPV_{n,X}(r_B) < 0$. $NPV_{n,X}(r)$ is an increasing function and $NPV_{n,X}(r_B) < 0 = NPV_{nX}(r_A)$, therefore $r_B < r_A$.

It has been already shown that $r_B = \frac{PR}{P-R}$ is an increasing function of X and r_A is always greater than r_B, but these two facts do not guarantee that r_A is an increasing function of X. Figure 6 shows an example which satisfies our result so far, yet r_A is not an increasing function of X. The following corollary proves that r_A is an increasing function of the compensating balance X, hence such case as Figure 6 *never* takes place.

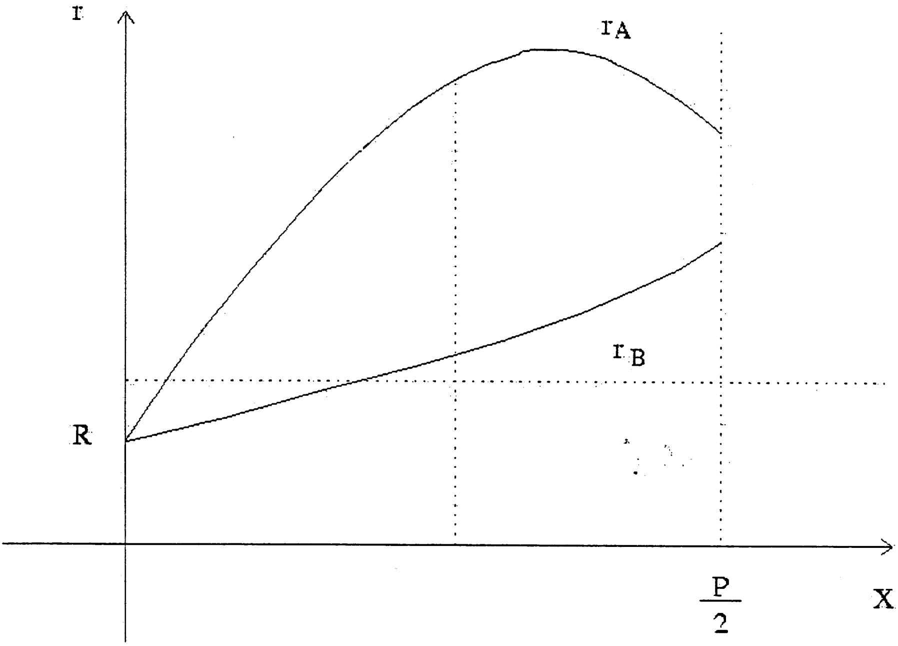

Fig. 6: r_A is greater than r_B, but not anincreasing function of X

Corollary r_A is an increasing function of X.

Proof Recall that $(P\text{-}X) = A_n f_n(r_A) - \frac{X}{(1+r_A)^n}$ where $A_n = \frac{P}{f_n(R)}$. Because $f_n(x)$ is decreasing, $NPV(n,X,r) = (P\text{-}X) - A_n f_n(r_A) + \frac{X}{(1+r_A)^n}$ is an increasing function of r for fixed n and X.

It suffices to show $X_1 < X_2$ implies $r_1 < r_2$ where r_1 and r_2 satisfy $NPV(n,X_1, r_1) = NPV(n,X_2, r_2) = 0$.

By definition

$$NPV(n,X,r_2) = (P\text{-}X_1) - A_n f_n(r_2) + \frac{X_1}{(1+r_2)^n} \tag{16}$$

and

$$NPV(n,X_2,r_2) = (P-X_2) - A_n f_n(r_2) + \frac{X_2}{(1+r_2)^n} \tag{17}$$

Therefore

$$\begin{aligned} NPV(n,X_1,r_2) - NPV(n,X_2,r_2)(P-X_1)(P-X_2) &+ \frac{1_1}{(1+r_2)^n}(X_1-X_2) \\ &= -(X_1-X_2) + \frac{1}{(1+r_2)^n}(X_2-X_1) \\ &= \left\{\frac{1}{(1+r_2)^n} - 1\right\}(X_1-X_2) = \left\{1 - \frac{1}{(1+r_2)^n}\right\}(X_2-X_1) \end{aligned} \tag{18}$$

Both $\left\{1 - \frac{1}{(1+r_2)^n}\right\}$ and (X_2-X_1) are positive implies $NPV(n,X_1,r_2) > NPV(n,X_1,r_1)$ implies $r_2 > r_1$ because NPV(n,X,r) is an increasing function of r for fixed n and X, proving the corollary.

4. Conclusion

When a compensating balance is not imposed, the effective costs of debt for the two different payment methods, full amortization method and bullet loan method are the same as the nominal interest rate. When a compensating balance is imposed, their respective effective costs of debt become different. Both effective costs increase as the newly imposed compensating balance increases. This paper provided a mathematical proof that the true cost of a fully amortized loan is always greater than that of a bullet loan. Our findings lead to the following conclusion: whenever a bank imposes a compensating balance, the borrowing firm should prefer a bullet loan to a fully-amortized one to avoid an ambush posed by such a balance.

Appendix

A. The Present Value Interest Factor for an Ordinary Annuity

For $0 < x < 1$, $fn(x) = \frac{1}{x}\{1 - \frac{1}{(1+x)^n}\}$ is decreasing in x and $g_n(x) = 1 - \frac{1}{(1+x)^n}$ is increasing in x.

Proof

$$\begin{aligned} \frac{df_n(x)}{dx} &= -\frac{1}{x^2}\{1-\frac{1}{(1+x)^n}\} + \frac{1}{x}\{0-\frac{n}{(1+x)^{n+1}}\} \\ &= -\frac{1}{x^2}\,\frac{(1+x)^n - 1}{(1+x)^n} + \frac{1}{x}\,\frac{n}{(1+x)^{n+1}} \\ &= -\frac{(1+x)^{n+1} + (n+1)\,x+1}{x^2(1+x)^{n+1}} \end{aligned} \tag{A.1}$$

Let $h(x) = -(1+x)^{n+1} + (n+1)x+1$, $(0<x<1)$, then

$$h'(x) = -(n+1)(1+x)^n + (n+1) + 0$$

$$= (n+1)\{1-(1+x)n\} < 0 \text{ for } 0<x<1 \text{ and} \tag{A.2}$$

$$h(0) = -(1+0)^{n+1} + (n+1)\,0 + 1 = -1+0+1 = 0. \tag{A.3}$$

$h(x)$ is a decreasing function on $(0,1)$ and $h(0)=0$, hence for $0 < x < 1$, hence $h(x)<0$.

Therefore $\frac{df_n(x)}{dx} = \frac{h(x)}{x^2(1+x)^{n+1}} < 0$ and $f_n(x)$ is a decreasing function.

$\frac{dg_n(x)}{dx} = 0 + \frac{n(1+x)^{n-1}}{(1+x)^{2n}} = \frac{n}{(1+x)^{n+1}} > 0$ for all $0 < x < 1$, hence $g(x)$ is increasing.

B. *The IRR* $r_A = (P\text{-}X) - A_n f_n(r) + \frac{X}{(1+r)^n}$

For a fixed n and X, $NPV_{n,X}(r)$ is an increasing function of r.

Proof. Write $X = (1+\alpha)A_n$ for $-1 < \alpha < \frac{f_n(R)}{2} - 1$. This range of decides the lower bound and upper bound of X so that $0 < X < \frac{P}{2}$. Then

$$\frac{d}{dr} NPV_{n,X}(r) = -A_n f_n(r) - X\frac{n}{(1+r)^n} \tag{B.1}$$

and this implies

$$\frac{d}{dr} NPV_{n,X}(r) = -A_n \frac{-(1+r)^{n+1} + (n+1)r+1}{r^2(1+r)^{n+1}} - (1+\alpha)A_n \frac{n}{(1+r)^{n+1}}$$

$$= A_n\left\{ \frac{-(1+r)^{n+1} - (n+1)r-1}{r^2(1+r)^{n+1}} - (1+\alpha)\frac{n}{(1+r)^{n+1}}\right\} \tag{B.2}$$

$$= A_n\left\{\frac{(1+r)^{n+1} - (n+1)r-1-(1+\alpha)nr^2}{r^2(1+r)^{n+1}}\right\}$$

Hence $NPV_n'(r) > 0$ if and only if $\left\{\frac{(1+r)^{n+1} - (n+1)r-1-(1+\alpha)nr^2}{r^2(1+r)^{n+1}}\right\} > 0$, which again implies $(1+r)^{n+1} - (n+1)r-1-(1+\alpha)nr^2 > 0$.

Using the binomial formula $(1+r)^{n+1} = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}x^k$

rewrite $(1+r)^{n+1} = 1+(n+1)r + \frac{(n+1)n}{2} r^2 + F(r)$, then

$$\begin{aligned}(1+r)^{n+1} - (n+1)r-1-(1+\alpha)nr^2 &= 1 + (n+1)r + \frac{(n+1)n}{2} r^2 + F(r) \\ &\quad -(n+1)r-1-(1+\alpha)nr^2 \\ &= \frac{n}{2} r^2 \{n-1-2\alpha\}+F(r) \\ &= \frac{n}{2} r^2 \{n-1+2-f_n(R)\}+F(r) \\ &= \frac{n}{2} r^2 \{n+1-f_n(R)\}+F(r).\end{aligned} \tag{B.3}$$

$f_n(R)$ is a decreasing function (Part A of this Appendix) and

$$\begin{aligned}\lim_{R\to 0} f_n(r) = \lim_{R\to 0} \frac{(1+R)^{n-1}-1}{R(1+R)^n} &= \lim_{R\to 0} \frac{n(1+R)^{n-1}-0}{1(1+R)^n+Rn(1+R)^{n-1}} \\ &= \frac{n(1+0)^{n-1}}{1(1+0)^n+0n(1+0)^{n-1}} = \frac{n}{1} = n.\end{aligned} \tag{B.4}$$

Hence (B.3) is greater than $\frac{nr^2}{2}\{n+1-n\} +F(r) = \frac{nr^2}{2} +F(r) >0$ because $F(r)$ is a polynomial in r with positive coefficients. This proves $NPV'_{n,X}(r) > 0$, therefore NPVn,X(r) is increasing.

C. Open Question and Conjecture

While working on the relation between the compensating balance and effective costs of debt, the authors became interested in another side of the effective costs of debt: when the compensating balance remains the same, are the effective costs of debt affected by the length of the loan period? Since we already know that the effective cost of debt under the bullet loan method depends only on the principal, the nominal interest rate and the compensating balance, our question boils down to the case of a full amortization payment method. From numerical experiments, the authors found the same pattern in effective cost of debt for other principal and loan period under full amortization method, which are summarized in the following conjecture.

Conjecture. There exists a loan period N_0 for which the effective cost of debt r_A for full amortization method is the maximum. r_A increases as the loan period n increases up to N_0, then decreases as n increases past N_0 and approaches PR/(P-X) as n grows to ∞.

The authors have not been able to present a complete proof of this conjecture but can provide some facts about the NPV function when the compensating balance is fixed.

Recall that $NPV(n,X,r)=(P-X)-A_n f_n(r_A) = \frac{X}{(1+r_A)^n}$ where $A_n = \frac{P}{R}\{1-\frac{1}{(1+R)^n}\}$ and $f_n(r)= \frac{1}{r}\{1-\frac{1}{(1+r)^n}\}$ is an increasing function of r and the effective cost of debt r_A is the value of r such that $NPV(r_A,X,n)=0$. For a fixed compensating balance X, NPV(,X,n) is a function of r and n and we write this as $NPV_X(r,n)$. Furthermore, for a fixed parameter $\rho > R$, $NPV_X(,n)$ is a function of n only. We can extend the domain of $NPV_X(\rho,n)$ to the set of all nonnegative real numbers so that $NPV_X(\rho,n)$ is a continuous function of n. For each n, there will be an r_A such that $NPV_X(r_A,n) = 0$, hence $NPV_X(\rho, n) > 0$ if $\rho > r_A$ and $NPV_X(\rho,n) < 0$ if $< r_A$. Then we can prove the following Lemma.

Lemma. For $NPV_X(\rho,n)$, there exists n_0 (not necessarily an integer) such that .

$\frac{d}{dn} NPV_n(\rho,n) < 0$ for $n < n_0$ and $\frac{d}{dn} NPV_n(\rho\rho n) > 0$ for $n > n_0$.

As long as it can be proved that the above n_0 is independent of the parameter , one can use the following duality of r_A and NPV_X: it suffices to show that $r_A = r_A(n)$ is increasing(decreasing, respectively) when $NPV_X(,n)$ is decreasing (increasing, respectively). Then $r_A(n)$ has its maximum value at $n = n_0$ where n_0 is the value decided by the previous lemma such that $NPV_X(\rho,n_0) \leq NPV_X(\rho,n)$ for all positive real number n.

The result is illustrated in Figure 7: the upper figure shows the result of Corollary and the lower picture is a gist of Lemma.

We prove by contradiction. Suppose there is a pair $n_1 < n_2 < n_0$ such that $NPV_X(\rho,n_1) > NPV_X(\rho,n_2)$ and $r_A(n_1) > r_A(n_2)$. By the previous Lemma, $NPV_X\rho(,n)$ is decreasing regardless of the choice of ρ for $n < n_0$, so choose ρ such that $r_A(n_1) > \rho > r_A(n_2)$. Then

$$NPV_X(r_A(n_1),n_1) = 0 > NPV_X(\rho,n_1) \text{ and}$$

$$NPV_X(r_A(n_2),n_2) = 0 > NPV_X(\rho,n_2) \text{ hence}$$

$$NPV_X(\rho,n_1) < 0 < NPV_X(\rho,n_2) \text{ contradicting our assumption.}$$

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[1] A term loan extends 1 year and up to 15 years, with the most common maturities falling between 1 and 5 years. This is the most common form of intermediate bank loan to commercial enterprises. It is appropriate to finance inventory, permanent working capital needs, or plant and equipment. The loan repayment schedule may require monthly, quarterly, semiannual, or annual payment. Johnson ([9]).

[2] For a three-year loan with $8,000 principal at a stated interest rate of 12.50% per year, the effective cost of debt is 12.50%.

[3] Descartes' Rule of Signs: when a polynomial $f(x)=a_0+a_1x+...a_nx^n$ has k sign changes of the coefficients, the equation can have up to k zeros.

Therefore for any $n_1 < n_2 < n_0$ and $n_0 < n_3 < n_4$,

$$r_A(n_1) < (n_2) \text{ and } NPV_X(\rho,n_1) > NPV_X(\rho,n_2) \tag{C.3}$$

and for the same reason

$$r_A(n_3) < r_A(n_4) \text{ and } NPV_X(\rho,n_3) > NPV_X(\rho,n_4) \tag{C.4}$$

Then

$$NPV_X\, r_A(n),n) = (P\text{-}X)\text{-}\ \frac{PR}{1\text{-}(1+R)^{-n}}\frac{1\text{-}(1+r_A)^{-n}}{r_A} + \frac{X}{(1+r_A)} = 0, \text{ hence}$$

$$\lim_{n\to\infty} NPV_X\, (r_A(n),n) = (P\text{-}X)\text{-}\ \frac{PR}{1\text{-}0}\frac{1\text{-}0}{r_A} + X.0 = 0 \text{ and this implies}$$

$$\lim_{n\to\infty} r_A = \frac{PR}{P\text{-}X}. \tag{C.5}$$

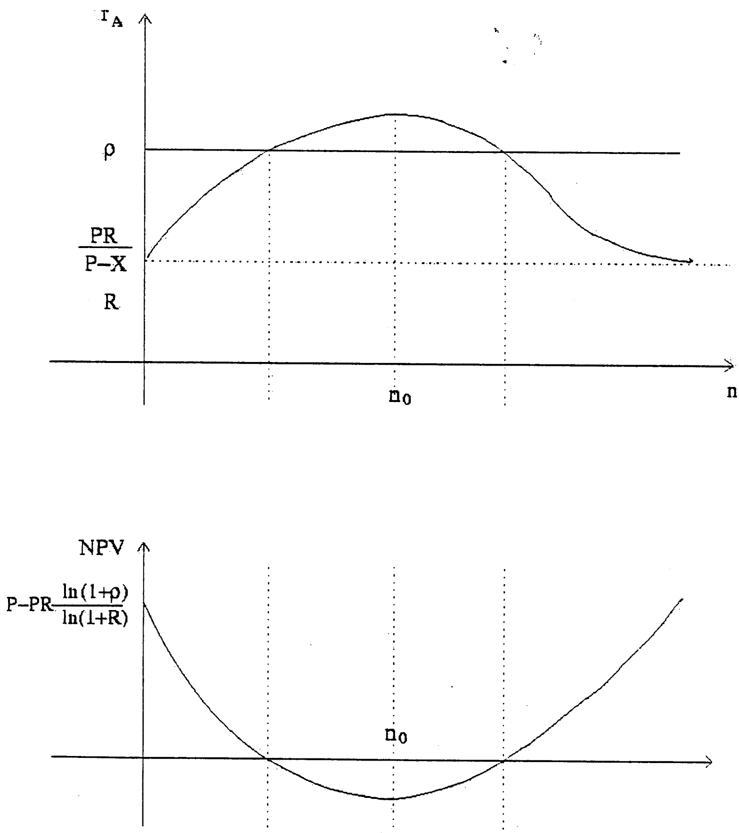

Fig. 7: Effective cost of debt and net present value vs. loan period

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